COMBINATORICA
Bolyai Society – Springer-Verlag

Combinatorica **25** (5) (2005) 503–518

SMALL FORBIDDEN CONFIGURATIONS IV: THE 3 ROWED CASE

RICHARD ANSTEE*, ATTILA SALI†

Received October 25, 2002

The present paper continues the work begun by Anstee, Griggs and Sali on small forbidden configurations. We define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Let F be a $k{\times}l$ (0,1)-matrix (the forbidden configuration). Small refers to the size of k and in this paper $k=3$. Assume A is an $m\times n$ simple matrix which has no submatrix which is a row and column permutation of F. We define $\text{forb}(m,F)$ as the best possible upper bound on n, for such a matrix A, which depends on m and F. We complete the classification for all 3-rowed (0,1)-matrices of $\text{forb}(m,F)$ as either $\Theta(m)$, $\Theta(m^2)$ or $\Theta(m^3)$ (with constants depending on F).

1. Introduction

The study of forbidden configurations is a problem in extremal set theory. The language we use here is matrix theory which conveniently encodes the problems. We define a *simple* matrix as a (0,1)-matrix with no repeated columns. Such an $m{\times}n$ simple matrix can be thought of a set of n subsets of $\{1,2,\dots,m\}$ with the rows indexing the elements and the columns indexing the subsets. Assume we are given a $k{\times}l$ (0,1)-matrix F. We say that a matrix A has a *configuration* F if a submatrix of A is a row and column permutation of F and so F is referred to as a *configuration* of A (sometimes called *trace*).

Mathematics Subject Classification (2000): 05D05; 05C65, 05C35

* Research is supported in part by NSERC.

† Research was done while the second author visited the University of British Columbia supported by NSERC of first author. Research was partially supported by Hungarian National Research Fund (OTKA) numbers T034702 and T037846.

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We often blur the distinction between a matrix and the related equivalence class of matrices under arbitrary row and column permutations which can be called a *configuration*. We say that a matrix A has no *configuration* F if no submatrix of A is a row and column permutation of F and so F is referred to as a *forbidden configuration* (of A). A variety of combinatorial objects can be defined using forbidden configurations.

We define $\text{forb}(m,F)$ as the smallest value (depending on m,F) so that if A is a simple $m\times n$ matrix and A has no configuration F then $n \le \text{forb}(m,F)$. Alternatively $\text{forb}(m,F)$ is the smallest value so that if A is a simple $m\times(\text{forb}(m,F)+1)$ matrix then A must have a configuration F. Many results have been obtained about $\text{forb}(m,F)$ [2,3,5].

At this point all values known for $\text{forb}(m,F)$ can be expressed as $\Theta(m^e)$ for some integer e. We have completed the classification for $2\times l$ matrices F in [2] and we now complete the classification for $3\times l$ matrices F in this paper. We also are willing to put forward a conjecture on what properties of F drive the exponent e. Roughly speaking, we propose a set of constructions and guess that these constructions are sufficient to deduce the exponent e in the expression $\Theta(m^e)$.

We use the notation K_k to denote the $k\times 2^k$ simple matrix of all possible columns on k rows. The basic result for $\text{forb}(m,F)$ is as follows.

Theorem 1.1 (Sauer [9], Perles and Shelah [10], Vapnik and Chervonenkis [11]). *We have that* $\text{forb}(m,K_k)=\binom{m}{k-1}+\binom{m}{k-2}+\cdots+\binom{m}{0}$ *and so* $\text{forb}(m,K_k)$ *is* $\Theta(m^{k-1})$.

The asymptotic growth of $\Theta(m^{k-1})$ was what interested Vapnik and Chervonenkis. Füredi [8] noted the following general bound.

Theorem 1.2 ([8]). *Let F be a $k\times l$ (0,1)-matrix. We have that* $\text{forb}(m,F)$ *is* $O(m^k)$.

We make a few simple observations. If we let A^c denote the 0-1-complement of A then $\text{forb}(m,F^c)=\text{forb}(m,F)$. Also, if F' is a row and column permutation of a submatrix of F (i.e. F has a configuration F'), then $\text{forb}(m,F)\ge\text{forb}(m,F')$. Some notations help us describe the most important matrices. Let K_k^s denote the $k\times\binom{k}{s}$ simple matrix of all possible columns of column sum s. The following configurations are needed for Theorem 1.3.

$$F_1=\begin{bmatrix}1\\0\\0\end{bmatrix}\qquad F_2=\begin{bmatrix}1&0&1&0\\0&1&1&1\\0&0&0&1\end{bmatrix}\qquad F_3=\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}$$

$$F_4(t) = \left[\begin{array}{cccccccc} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 0 & \overbrace{1\cdots1}^{t} & 1 & \overbrace{0\cdots0}^{t} & 1 \\ 0 & 0\cdots0 & 1\cdots1 & 0 & 1\cdots1 & 0 & 1\cdots1 & 1 \\ 0 & 0\cdots0 & 0\cdots0 & 1 & 0\cdots0 & 1 & 1\cdots1 & 1 \end{array}\right]$$

$$F_5(t) = \left[\begin{array}{cccccccc} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 0 & 1 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & 1 \\ 0 & 0\cdots0 & 1\cdots1 & 0 & 1 & 0\cdots0 & 1\cdots1 & 1 \\ 0 & 0\cdots0 & 0\cdots0 & 1 & 0 & 1\cdots1 & 1\cdots1 & 1 \end{array}\right]$$

$$F_6(t) = \left[\begin{array}{cccccc} 0 & \overbrace{1\cdots1}^{t} & \overbrace{0\cdots0}^{t} & \overbrace{0\cdots0}^{t} & \overbrace{1\cdots1}^{t} & \overbrace{1\cdots1}^{t} \\ 0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 & 0\cdots0 \\ 0 & 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1 \end{array}\right]$$

We use the notation $[A|B]$ to denote the matrix obtained from concatenating the two matrices A and B. We use the notation $k\cdot A$ to denote the matrix $[A|A|\cdots|A]$ consisting of k copies of A concatenated together. We give precedence to the operation $\cdot$ (multiplication) over concatenation so that for example $[2\cdot A|B]$ is the matrix consisting of the concatenation of B with the concatenation of two copies of A.

Theorem 1.3. *Let F be a $3\times l$ (0,1)-matrix.*
(Linear Cases) If F has at least one column and if F is a configuration in F_2 then $\text{forb}(m,F)=\Theta(m)$.
(Quadratic Cases) If F has at least one configuration from K_3^0, K_3^1, K_3^2, K_3^3, $2\cdot F_1$, $2\cdot F_1^c$ or F_3 and if F is a configuration in $F_4(t)$, $F_5(t)$, $F_6(t)$ or $F_6(t)^c$ for some $t>0$, then $\text{forb}(m,F)=\Theta(m^2)$.
(Cubic Cases) If F has at least one configuration from $2\cdot K_3^0$, $[2\cdot K_3^1|K_3^2]$, $[2\cdot K_3^1|K_3^3]$, $[K_3^0|2\cdot K_3^2]$, $[K_3^1|2\cdot K_3^2]$ or $2\cdot K_3^3$ then $\text{forb}(m,F)=\Theta(m^3)$.

Proof. The linear bound for $\text{forb}(m,F_2)$ is Theorem 3.3 in [2]. The quadratic bound for $\text{forb}(m,F_4(t))$ is Theorem 3.9 in [2]. The quadratic bound for $\text{forb}(m,F_5(t))$ is Theorem 4.2 in this paper and the quadratic bound for $\text{forb}(m,F_6(t))$ is Theorem 4.1 in this paper. The cubic bound for all 3-rowed F follows from Theorem 1.2 above. We had already shown that $\text{forb}(m,t{\cdot}K_3^1)$ was $O(M^{2\frac{1}{3}})$ in Theorem 4.1 [3] which was strong evidence for our Theorem 4.1 in this paper. All the lower bounds follow from the constructions given in Conjecture 1.4 below but are developed in [2]. Quadratic lower bounds for $\text{forb}(m,K_3^1)$, $\text{forb}(m,K_3^2)$, $\text{forb}(m,F_3)$ are in Corollary 3.5 [2], quadratic lower bound for $\text{forb}(m,K_3^3)$ (and hence $\text{forb}(m,K_3^0)$ by taking the 0-1-complement) is in Theorem 3.6 [2], quadratic lower bound for forb$(m, 2\cdot F_1)$ (and hence forb$(m, 2\cdot F_1^c)$) is in Theorem 3.7 [2]. A cubic lower bound for forb$(m, 2\cdot K_3^3)$ (and hence forb$(m, 2\cdot K_3^0)$) is in Theorem 3.9 [2] and cubic lower bounds for forb$(m, [2\cdot K_3^2|K_3^0])$ and forb$(m, [2\cdot K_3^2|K_3^1])$ (and hence also for forb$(m, [2\cdot K_3^1|K_3^3])$, forb$(m, [2\cdot K_3^1|K_3^2])$) are in Theorem 3.10 [2]. We have combined the various results, using the results about 0-1-complements, to obtain this classification and the reader can verify (with some effort) that every $3\times l$ (0,1)-matrix F is covered by one of the three cases. ∎

What structures in F drive this rather complicated classification? It is remarkable enough that forb(m, F) is $\Theta(m^e)$ for an integer e and not some other function.

Below is a conjecture that focused our attention while exploring results in this paper. Let A_i be an $m_i \times n_i$ simple matrix for $1 \le i \le k$. Denote $A_1 \times A_2 \times \cdots \times A_k$ as the $(\sum m_i) \times (\prod n_i)$ simple matrix whose columns are formed in all possible ways by putting a column of A_1 in the first m_1 rows and putting a column of A_2 in the next m_2 rows etc. Let T_h denote the $h\times h$ *triangular* matrix

$$T_h = \begin{bmatrix} 1 & & & 1's \\ & 1 & & \\ & & \ddots & \\ 0's & & & 1 \end{bmatrix}.$$

Let F be a $k\times l$ (0,1)-matrix. Let $X(F)$ be the smallest p so that F is a configuration in $A_1 \times A_2 \times \cdots \times A_p$ for every choice of A_i as either $K_{m/p}^1$, $K_{m/p}^{(m/p)-1}$ or $T_{m/p}$. We assume m is large and divisble by p, in particular that $m \ge (k+1)(kl+1)$ so that $m/p \ge kl+1$. Divisibility by p does not affect the asymptotics since we can use a simple submatrix of a simple matrix that avoids F for construction purposes. We are using the fact that we need only consider p-fold products for $p \le k+1$, since we can find F as a configuration in $A_1 \times A_2 \times \cdots \times A_{k+1}$ by taking 1 row from each of the first k products (each row has [01]) and then, since we are taking zero rows from the final A_{k+1}, we get the configuration $(m/(k+1))\cdot K_k$ in the product and F is a configuration in $l\cdot K_k$.

If F is a configuration in the p-fold product $A_1 \times A_2 \times \cdots \times A_p$, assume that a_i rows of A_i are used with $\sum_{i=1}^p a_i = k$. If we form the submatrix of A_i of a_i rows, then we would be interested in at most l copies of a given column on these rows (F has l columns) if this is possible. Now for $t \ge k+l$, any a_i rows of K_t^1 contains l columns of 0's as well as a copy of $K_{a_i}^1$. The analogous result is true for K_t^{t-1}. Also for $t \ge kl+l$, the a_i rows of T_t consisting of rows $l+1, 2l+1, 3l+1, \ldots, kl+1$ have l columns of 0's and $l\cdot T_{a_i}$. Thus as long as $m \geq (k+1)(kl+1)$ we are able to use the matrices A_i as if they were arbitrarily large.

Note that the definition of $X(F)$ ensures forb(m,F) is $\Omega(m^{X(F)-1})$, although for $X(F)=1$ a little care must be taken.

One can find in [3] that F_2 is contained in $T_{m/2} \times T_{m/2}$ whereas F_3 is not a configuration in $T_{m/2} \times T_{m/2}$. This is critical in computing $X(F_2)=2$ and $X(F_3)=3$. Interestingly, F_2^c is F_2 and F_3^c is F_3 when viewed as configurations.

To compute $X(F_6(t))=3$, we find that $F_6(t)$ is contained in every three fold product $A_1 \times A_2 \times A_3$ by selecting one row from each of the terms with the one exception of $K^1_{m/3} \times K^1_{m/3} \times K^{m/3-1}_{m/3}$, where we can find $F_6(t)$ by taking two rows from the first $K^1_{m/3}$ and one row from the second $K^1_{m/3}$ and none from the rows from $K^{m/3-1}_{m/3}$. Also $F_6(t)$ is not a configuration in $K^1_{m/2} \times K^1_{m/2}$.

Computing $X(F)$ is non trivial and we have yet to make a direct connection of our proofs of asymptotic bounds for forb(m,F) with the derivation of $X(F)$.

Conjecture 1.4.

$$\text{forb}(m,F) = \Theta(m^{X(F)-1}).$$

Our results show this conjecture is true for F with either 1,2 or 3 rows. We do wish the reader to draw the analogy with the Erdős Stone result [7] on the maximum number of edges in a graph with a forbidden subgraph where asymptotic bounds are determined by the chromatic number $\chi(G)$ of the forbidden subgraph G. Our asymptotics are weaker in that we do not predict the constant implicit in the notation $\Theta(m^p)$. We also point out that the result of Balogh and Bollobás [6] that if we have a simple $m \times n$ matrix and we forbid the triple of configurations K_k^1, K_k^{k-1}, T_k, then n can be bounded by a constant, (independent of m). This coincides with our conjecture and provides further evidence that the three matrices K_k^1, K_k^{k-1}, T_k are relevant in considering forbidden configurations.

We hope our proof techniques developed in this paper will assist us in tackling this conjecture in the future. As an example, we would need arguments for the forbidden configuration:

$$F_7(t) = \begin{bmatrix} \overbrace{11\cdots1}^{t} & \overbrace{00\cdots0}^{t} \\ 11\cdots1 & 00\cdots0 \\ 00\cdots0 & 11\cdots1 \\ 00\cdots0 & 11\cdots1 \end{bmatrix}$$

which, if the conjecture was true, has $\mathrm{forb}(m, F_7(t))$ being $O(m^2)$ since we know $\mathrm{forb}(m, F_7(2))$ is $O(m^2)$ from [1]. The conjecture predicts that the asymptotic bound for a matrix with two copies of a column is the same as the asymptotics for a matrix with many copies of that column. This follows from our discussion above where we have taken a_i columns from A_i and we either have at least l copies of a column or 1 copy of a column on the a_i rows. The cross product preserves this property, even when the cross product involves a factor with $a_i = 0$ since in that case all columns appear at least l times.

2. What is missing if a configuration F is avoided?

A careful consideration is required to see what is missing from A when either $F_5(t)$ or $F_6(t)$ is not a configuration in A. We wish to use the following terminology. Let $\{i,j,k\}$ be a triple of rows of a matrix $A = (a_{rs})$. We say that we have

$$\text{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{1}$$

if in every column q of A we do not have $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occurring. As well, we say that there are

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{2}$$

if there are at most $t-1$ columns q of A in which $a_{iq} = d, a_{jq} = e$ and $a_{kq} = f$ all occur.

Let S_3 denote the symmetric group on three symbols.

Proposition 2.1. *Let A be a (0,1)-matrix with no configuration $F_6(t)$. Let a,b,c be a triple of rows of A. Then we either have a permutation $\pi_1 \in S_3$ with $\pi_1(a) = i$, $\pi_1(b) = j$, $\pi_1(c) = k$ (note that $\{a,b,c\}$ and $\{i,j,k\}$ are the same as sets) with*

$$\textit{no}\ \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{3}$$

or if we do not have (3), then we have a permutation $\pi_2 \in S_3$ *with* $\pi_2(a)=i$, $\pi_2(b)=j$, $\pi_2(c)=k$ *with*

$$\text{(4)} \qquad \textit{at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

or if we do not have (3), (4), then we have a permutation $\pi_3 \in S_3$ *with* $\pi_3(a)=i$, $\pi_3(b)=j$, $\pi_3(c)=k$ *with*

$$\text{(5)} \qquad \textit{at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \textit{and at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Proof. If one of (3), (4), (5) is true we have no $F_6(t)$. Give (3) is false, we either have $t \cdot K_3^1$ in the triple of rows or not. If not, then (4) holds for some ordering. If we do have $t \cdot K_3^1$ in the triple of rows, then t copies of two columns of two 1's (in the triple of rows) will yield $F_6(t)$ and so at most one column of two 1's appears t or more times. Thus (5) holds. ∎

The proof of Proposition 2.1 is relatively easy but the result for $F_5(t)$ is more subtle.

Proposition 2.2. *Let A be a (0,1)-matrix with no configuration* $F_5(t)$*. Let* a, b, c *be a triple of rows of A. Then we either have a permutation* $\pi_1 \in S_3$ *with* $\pi_1(a)=i$, $\pi_1(b)=j$, $\pi_1(c)=k$ *with*

$$\text{(6)} \qquad \textit{no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \textit{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \textit{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \textit{ or no } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

or if we do not have (6), then we have a permutation $\pi_2 \in S_3$ *with* $\pi_2(a)=i$, $\pi_2(b)=j$, $\pi_2(c)=k$ *with*

$$\text{(7)} \qquad \textit{at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \textit{and at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

or if we do not have (6), (7), then we have a permutation $\pi_3 \in S_3$ *with* $\pi_3(a)=i$, $\pi_3(b)=j$, $\pi_3(c)=k$ *with*

$$\text{(8)} \qquad \textit{at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \textit{and at most } t-1 \textit{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

or if we do not have (6), (7), (8), then we have a permutation $\pi_4 \in S_3$ *with* $\pi_4(a)=i$, $\pi_4(b)=j$, $\pi_4(c)=k$ *with*

$$\text{at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and at most } t-1 \text{ of } \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{9}$$

Proof. If one of (6), (7) or (8) is true we have no $F_5(t)$. Given (6), (7) and (8) are false, then the submatrix of A formed by rows $\{a,b,c\}$ has every possible column on three rows at least once and at least t copies of 2 columns of one 1 and has at least t copies of 2 columns of two 1's. Thus either we have $F_4(t)$ or $F_5(t)$. If we have $F_4(t)$ (and not $F_5(t)$), we note that the two columns of one or two 1's that appear at most $t-1$ times are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

whereas for $F_5(t)$ the two columns of one or two 1's that appear at most $t-1$ times are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus (9) holds if we have $F_4(t)$ and not $F_5(t)$. ∎

3. Implications

We will use the notation

$$ij \rightarrow_0 k, \tag{10}$$

where i,j,k are row indices, and refer to this as a 0-implication. It is not required that i,j,k be distinct. We say that a column q of a matrix $A=(a_{rs})$ *violates* the 0-implication $ij \rightarrow_0 k$ if we have $a_{iq}=0$, $a_{jq}=0$, and $a_{kq}=1$. We say that a column q satisfies the 0-implication if the column does not violate the 0-implication, namely when $a_{iq}=0$, $a_{jq}=0$, we have $a_{kq}=0$.

If the matrix A has

$$\text{at most } t-1 \quad \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

then the 0-implication $ij \rightarrow_0 k$ is violated by at most $t-1$ columns of A.

We can also refer by analogy to 1-implications using the notation

$$ij \to_1 k$$

by interchanging the roles of 0 and 1. Again, if the matrix A has

$$\text{at most } t-1 \quad \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

then the 1-implication $ij \to_1 k$ is violated at most $t-1$ times.

Let $\mathrm{Imp}_0(A)$ denote the 0-implications $ij \to_0 k$ which are violated at most $t-1$ times by columns of A. We form a directed graph $D_0(A)$ from $\mathrm{Imp}_0(A)$ with node set equal to all $\binom{m}{2}$ pairs of rows. We have an arc $ij \to kl$ in D_0 if and only if $ij \to_0 k$ and $ij \to_0 l$ are in $\mathrm{Imp}_0(A)$. We can use $ij \to_0 i$ and $ij \to_0 j$ as 0-implications that are never violated and so $ij \to ik$ and $ij \to jk$ follow from $ij \to_0 k$.

Since $D_0(A)$ is a directed graph, we can apply the standard decomposition and topological sort to identify the strongly connected components and also order the nodes of $D_0(A)$ so that if $ij \to kl$ in $D_0(A)$, then either kl appears later in the ordering or ij and kl belong to the same strongly connected component.

Let $C(ij)$ denote the strongly connected component of $D_0(A)$ containing the node ij. We define the support of a strongly connected component C as

$$\mathrm{supp}(C) = \bigcup_{ij \in C} \{i, j\}$$

Our goal is to select a small subset of the implications of $\mathrm{Imp}_0(A)$ (preferably of size $O(m^2)$) so that a column that violates one of the implications also violates one of the chosen implications. A pigeonhole principle then ensures that the number of columns with violations is at most $t-1$ times number of chosen implications (and hence $O(m^2)$ if we were lucky). The motivation for implications is partly in Theorem 2.2 in [3] and partly from *functional dependencies* in database theory.

The following classification was very useful. An implication $ij \to_0 k$ of $\mathrm{Imp}_0(A)$ is called an *outside* implication if $k \notin \mathrm{supp}(C(ij))$ and an implication $ij \to_0 k$ of $\mathrm{Imp}_0(A)$ is called an *inside* implication if $k \in \mathrm{supp}(C(ij))$.

Lemma 3.1. *We can select $O(m^2)$ inside 0-implications from $\mathrm{Imp}_0(A)$ so that if an inside 0-implication in $\mathrm{Imp}_0(A)$ is violated then one of the selected 0-implications is violated.*

Proof. We define a subset $\mathrm{Imp}_0'(A)$ of the inside implications $\mathrm{Imp}_0(A)$ where $\mathrm{Imp}_0'(A)$ consists of $O(m^2)$ inside implications. Inside each strongly connected component C of $D_0(A)$ on p nodes (p pairs of rows), we can find at most $2p-2$ arcs so that the directed graph consisting of the nodes of C and the up to $2p$–2 arcs results in a strongly connected graph. Thus we can select up to $2\binom{m}{2}-2$ arcs from those within strongly connected components to form a new directed graph $D_0'(A)$ which has the same strongly connected components as $D_0(A)$. If we have $ij\rightarrow kl$ in $D_0'(A)$, we can now form the inside implications in $\mathrm{Imp}_0'(A)$ by selecting from $\mathrm{Imp}_0(A)$ the inside implications $ij\rightarrow_0 k$ and $ij\rightarrow_0 l$. Thus $\mathrm{Imp}_0'(A)$ has at most $4\binom{m}{2}$ inside implications.

Imagine having a column q of A which violates an inside implication $ij\rightarrow_0 k$ with $k\in\mathrm{supp}(C(ij))$. Thus we have $a_{iq}=0$, $a_{jq}=0$ and $a_{kq}=1$. Now with $k\in\mathrm{supp}(C(ij))$, there must be some row l with $kl\in C(ij)$ and hence there is a directed path in $D_0'(A)$

$$ij = u_1v_1 \rightarrow u_2v_2 \rightarrow u_3v_3 \rightarrow \cdots \rightarrow u_qv_q = kl$$

But then $\mathrm{Imp}_0'(A)$ contains the implications

$$ij \rightarrow_0 u_2, \quad ij \rightarrow_0 v_2, \quad u_2v_2 \rightarrow_0 u_3, \quad u_2v_2 \rightarrow_0 v_3, \quad \ldots, \quad u_{q-1}v_{q-1} \rightarrow_0 k$$

With $a_{iq} = 0$, $a_{jq} = 0$ and $a_{kq} = 1$, we deduce that some implication of $\mathrm{Imp}_0'(A)$ is violated by column q. ∎

Sometimes we have two 0-implications on a triple i,j,k and so more can be deduced.

Lemma 3.2. *If a triple i,j,k of rows of A has the property that two of the three possible 0-implications $ij\rightarrow_0 k$, $ik\rightarrow_0 j$, $jk\rightarrow_0 i$ are in $\mathrm{Imp}_0(A)$, then these two 0-implications are inside 0-implications.*

Proof. Assume $ij\rightarrow_0 k$. Thus on the triple i,j,k we may assume without loss of generality that $ik\rightarrow_0 j$. Now $ij\rightarrow_0 k$ yields $ij\rightarrow ik$ in $D_0(A)$. Also $ik\rightarrow_0 j$ yields $ik\rightarrow ij$ in $D_0(A)$. Thus $ik\in C(ij)$ and so $k\in\mathrm{supp}(C(ij))$ and so $ij\rightarrow_0 k$ is an inside implication. ∎

We can do certain reductions on outside implications that we summarize in what follows.

Lemma 3.3. *We can choose a subset $\mathrm{Imp}_0''(A)$ of $\mathrm{Imp}_0(A)$ so that every violation of a 0-implication in $\mathrm{Imp}_0(A)$ yields a violation of a 0-implication in $\mathrm{Imp}_0''(A)$ with the property that if we have outside 0-implications $ij\rightarrow_0 k$, $ij\rightarrow_0 l$ in $\mathrm{Imp}_0''(A)$, we do not have $ik\rightarrow_0 l\in\mathrm{Imp}_0(A)$ and if we have outside 0-implications $ij\rightarrow_0 k$, $ij\rightarrow_0 l$, $ij\rightarrow_0 h$ in $\mathrm{Imp}_0''(A)$, we do not have $kl\rightarrow_0 h\in\mathrm{Imp}_0(A)$.*

Proof. We rely on the topological ordering of the nodes of $D_0(A)$ which are the pairs of rows of A. We first do the reduction in Lemma 3.1. We start with $\mathrm{Imp}_0''(A)$ consisting of $\mathrm{Imp}_0'(A)$ plus all the outside implications in $\mathrm{Imp}_0(A)$. We successively reduce $\mathrm{Imp}_0''(A)$ by processing pairs of rows in the topological order as follows. For each pair ij, we delete as many outside implications from ij as possible while preserving the property that every violation of an implication in $\mathrm{Imp}_0(A)$ yields a violation of an implication in the new reduced set of implications $\mathrm{Imp}_0''(A)$. If we are processing outside implications from ij and we have $ij \to_0 k$, $ij \to_0 l$ in $\mathrm{Imp}_0''(A)$ and $ik \to_0 l$ in $\mathrm{Imp}_0(A)$, then we can delete $ij \to_0 l$. This is because a violation of $ij \to_0 l$ will either violate $ij \to_0 k$ or $ik \to_0 l$. Now $ij \to_0 k$ is in $\mathrm{Imp}_0''(A)$. Also ik is later in the topological ordering than ij (in view of the implication $ij \to_0 k$ which yields $ij \to ik$ in $D_0(A)$ and the fact that $k \notin \mathrm{supp}(C(ij))$ so we do not have ik in $C(ij)$). If $ik \to_0 l$ is currently in $\mathrm{Imp}_0''(A)$ we are done. We note that we have not yet have processed outside implications from ik so if $ik \to_0 l$ is not currently in $\mathrm{Imp}_0''(A)$ then it must be because $ik \to_0 l$ is an inside implication that was deleted using Lemma 3.1. Now we can use the argument in Lemma 3.1 to verify that some remaining inside implication in $\mathrm{Imp}_0'(A)$ is violated. Thus deleting $ij \to_0 l$ from $\mathrm{Imp}_0''(A)$ will preserve the property that every violation of an implication in $\mathrm{Imp}_0(A)$ yields a violation of an implication in the new reduced set of implications $\mathrm{Imp}_0''(A)$.

We proceed in an inductive way to delete implications while preserving the property that every violation of an implication in $\mathrm{Imp}_0(A)$ yields a violation of an implication in the new reduced set of implications $\mathrm{Imp}_0''(A)$.

In a similar fashion we can assume that if we have $ij \to_0 k$, $ij \to_0 l$, $ij \to_0 h$ in $\mathrm{Imp}_0''(A)$, and $kl \to_0 h$ in $\mathrm{Imp}_0(A)$, then we can delete $ij \to_0 h$. ∎

The ordering was crucial to the formation of $\mathrm{Imp}_0''(A)$. Other reductions are possible. For example we can show that for $k \notin \mathrm{supp}(C(ij))$ we need only keep one implication of the form $uv \to_0 k$ for $uv \in C(ij)$. We did not need this in our proofs.

4. Proofs of Quadratic Bounds

Theorem 4.1. *We have that* $\mathrm{forb}(m, F_6(t))$ *is* $O(m^2)$ *i.e.* $\mathrm{forb}(m, F_6(t)) \le c_6(t)m^2$.

Proof. Let t be given. Let A be a simple $m \times n$ matrix with no configuration $F_6(t)$. Use Proposition 2.1. Each triple of rows i, j, k either has (3) or a 0-implication or two 1-implications.

Using Lemma 3.2 on the 1-implications, we can select at most $2m^2$ 1-implications so that a column of A which has violations of 1-implications has violations of one of the at most $2m^2$ 1-implications. By the pigeonhole principle, there are at most $2(t-1)m^2$ columns in A which have violations of 1-implications. Delete these columns to form a new $m\times n'$ simple matrix A' with $n\le n'+2(t-1)m^2$, where A' has no $F_6(t)$. We can reapply Proposition 2.1 to A' to deduce perhaps additional 0-implications (additional 1-implications would not be generated given the precedence in Proposition 2.1). But now if we have (5), we have

$$\text{no}\ \begin{matrix} i \\ j \\ k \end{matrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\ \text{and no}\ \begin{matrix} i \\ j \\ k \end{matrix}\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

We reduce the 0-implications $\mathrm{Imp}_0(A')$ to $\mathrm{Imp}''_0(A')$ as described in Lemmas 3.1, 3.3 but in view of Lemma 3.1 we focus attention on outside 0-implications. Assume we have in $\mathrm{Imp}''_0(A')$ outside implications

$$ij\to_0 k,\quad ij\to_0 l,\quad k,l\notin \mathrm{supp}(C(ij)).$$

Consider the triple i,k,l. By Lemma 3.3, we do not have $ik\to_0 l$ or $il\to_0 k$. Thus if we have a 0-implication of the triple i,k,l, then it must be $kl\to_0 i$. If the triples i,k,l and j,k,l both have 0-implications then we have $kl\to_0 i$ and $kl\to_0 j$ which yields $kl\to ij$ in $D_0(A')$. Then $ij\to_0 k,$, $ij\to_0 l$ yields $ij\to kl$ in $D_0(A')$ and so $kl\in C(ij)$ contradicting $k,l\notin \mathrm{supp}(C(ij))$. Thus we may assume the triple j,k,l does not have a 0-implication (we can make a similar argument if the triple i,k,l has no 0-implication). Hence it either has no column of 0's (on rows j,k,l) or it has two 1-implications (on rows j,k,l). If we have two 1-implications, we may assume we have $jk\to_1 l$. Now we have

$$\text{at most } t-1 \text{ of}\ \begin{matrix} i \\ j \\ k \\ l \end{matrix}\begin{bmatrix}0\\0\\1\\ \ \end{bmatrix}\ \text{and at most } t-1 \text{ of}\ \begin{matrix} i \\ j \\ k \\ l \end{matrix}\begin{bmatrix}0\\0\\ \ \\1\end{bmatrix}\ \text{and no}\ \begin{matrix} i \\ j \\ k \\ l \end{matrix}\begin{bmatrix}\ \\1\\1\\0\end{bmatrix}$$

forces

$$\text{at most } t-1 \text{ of}\ \begin{matrix} i \\ j \\ k \\ l \end{matrix}\begin{bmatrix}0\\ \ \\1\\0\end{bmatrix}$$

(since for any column q of $A'=(a'_{rs})$, $a'_{lq}=0$ and $a'_{kq}=1$ forces $a'_{jq}=0$) which is the 0-implication $il\to_0 k$. But this must have been discovered while computing $\mathrm{Imp}_0(A')$. Now this again contradicts Lemma 3.3. Hence we deduce that neither triple i,k,l or j,k,l contains two 1-implications and so one of the triples, say i,k,l has no column of 0's.

When the outside implications from ij are $ij \to_0 v_1$, $ij \to_0 v_2$, $\cdots$,$ij \to_0 v_q$ we repeat the above argument to deduce that for each pair v_r, v_s (with $1 \le r < s \le q$) we have

$$\text{no} \quad \begin{matrix} i \\ v_r \\ v_s \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or no} \quad \begin{matrix} j \\ v_r \\ v_s \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus a column that violates one of the outside implications from ij will violate at least $q-1$ of the q outside implications, and hence at least $\frac{1}{2}$ of the implications. Thus at most $2(t-1)\binom{m}{2}$ columns of A' can violate outside 0-implications. Also at most $O(m^2)$ columns of A' can violate inside 0-implications by Lemma 3.1. The number of columns with no violations is at most $\mathrm{forb}(m, K_3)$ which is $O(m^2)$ by Theorem 1.1. Thus we have shown n' is $O(m^2)$ and so n is also $O(m^2)$. ∎

Theorem 4.2. *We have that* $\mathrm{forb}(m, F_5(t))$ *is* $O(m^2)$ *i.e.* $\mathrm{forb}(m, F_5(t)) \le c_5(t)m^2$.

Proof. Let t be given. Let A be a simple $m \times n$ matrix with no configuration $F_5(t)$. We use Proposition 2.2 to deduce the 0-implications from (7) to form $\mathrm{Imp}_0(A)$ and the 1-implications from (8) to form $\mathrm{Imp}_1(A)$ always have two 0- or two 1-implications on the selected triples of rows. We can apply Lemma 3.2 to see that all these 0- and 1-implications are inside implications and then use Lemma 3.1 to both $\mathrm{Imp}_0(A)$ and $\mathrm{Imp}_1(A)$ to obtain $\mathrm{Imp}_0'(A)$ and $\mathrm{Imp}_1'(A)$, each with only $O(m^2)$ implications, where a violation in $\mathrm{Imp}_0(A)$ or $\mathrm{Imp}_1(A)$ results in a violation in $\mathrm{Imp}_0'(A)$ or $\mathrm{Imp}_1'(A)$. Thus the number of columns violating 0-implications from (7) or 1-implications from (8) is at most $O(m^2)$.

The case (9) in Proposition 2.2 that has

$$\text{at most } t-1 \quad \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and at most } t-1 \quad \begin{matrix} i \\ j \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

will imply

$$\text{at most } 2t-2 \quad \begin{matrix} i \\ k \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which we can think of as a 0-implication $i \to_0 k$ analogous to the notation already introduced. Such a special implication can be violated at most $2t-2$

times. Obviously the number of such implications is at most m^2 and so the number of columns violating these implications is at most $(2t-2)m^2$.

The remaining columns of A do not violate any implications arising from (7), (8) and (9). Let A' denote the matrix formed by these columns. Thus for every triple of rows $\{a,b,c\}$, we have (6). Hence by Theorem 1.1, the number of remaining columns is $O(m^2)$. ∎

5. Generalizations

We do not have enough tools yet to solve the 4-rowed case and have given a difficult case as $F_7(t)$ in the Introduction. Nonetheless, the methods do solve some important cases. Let S_p denote the symmetric group on p symbols.

Theorem 5.1. *We have that* $\mathrm{forb}(m,[K_p^0|t\cdot K_p^1])$ *is* $\Theta(m^{p-1})$.

Proof. The lower bound $\mathrm{forb}(K_p^{p-1})$ being $\Omega(m^{p-1})$ follows from our product constructions.

Let F denote $[K_p^0|t\cdot K_p^1]$. Let A be a simple matrix with no F. We use analogous notation to that already introduced. Using the proof ideas of Proposition 2.1, we see that for any p rows $s_1,s_2,\ldots,s_p$ of A we either have a permutation $\pi_1\in S_p$ with $\pi_1(s_1)=r_1$, $\pi_1(s_2)=r_2$, $\ldots$, $\pi_1(s_p)=r_p$ (note that $\{s_1,s_2,\ldots,s_p\}$ and $\{r_1,r_2,\ldots,r_p\}$ are the same as sets) with

$$\text{(11)}\qquad \text{no}\quad \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_{p-1} \\ r_p \end{matrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

or if we do not have (11), we have a permutation $\pi_2\in S_p$ with $\pi_2(s_1)=r_1$, $\pi_2(s_2)=r_2,\ldots,\pi_2(s_p)=r_p$ with

$$\text{(12)}\qquad \text{at most } t-1 \text{ of}\quad \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_{p-1} \\ r_p \end{matrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

In this setting we generalize the 0-implications of (10) to 0-implications of the form

$$r_1r_2\cdots r_{p-1}\rightarrow_0 r_p.$$

We say that the 0-implication is violated if $a_{r_1q}=0, a_{r_2q}=0, \cdots, a_{r_{p-1}q}=0$ and $a_{r_pq}=1$. If at most $t-1$ columns of A violate such a 0-implication, then we add it to $\mathrm{Imp}_0(A)$. As before we form a directed graph $D_0(A)$ with node set equal to all $t-1$-sets of rows and we have an arc $b_1b_2\cdots b_{p-1} \to c_1c_2\cdots c_{p-1}$ if and only if $b_1b_2\cdots b_{p-1} \to_0 c_i$ for $i\in\{1,2,\dots,p-1\}$. We use the standard directed graph decomposition for $D_0(A)$ to identify strongly connected components and we classify implications as either *inside* or *outside* 0-implications.

The number of columns in A with no violations is at most $O(m^{p-1})$ by Theorem 1.1. We may apply the reductions of Lemma 3.1 for the inside implications to choose $O(m^{p-1})$ from the $O(m^p)$ possible inside implications so that if any inside implication is violated then one of the chosen inside implications is violated. Apply the reduction of Lemma 3.3 to reduce $\mathrm{Imp}_0(A)$ to $\mathrm{Imp}''_0(A)$. Imagine we have many outside implications in $\mathrm{Imp}''_0(A)$ from a $p-1$ set $\{r_1,r_2,\dots,r_{p-1}\}$, say $r_1r_2\cdots r_{p-1}\to_0 t_i$ for $i=1,2,\dots,v$. We can use Lemma 3.3 to deduce that 0-implications of the form $t_{i_1}t_{i_2}\cdots t_{i_{p-1}}\to_0 t_{i_p}$ do not occur in $\mathrm{Imp}''_0(A)$. Thus every subset of p rows chosen from $\{t_1,t_2,\dots,t_v\}$ must have the column of p 0's forbidden as in (11). We deduce that every column has at least $v-p+1$ 1's on rows $t_1,t_2,\dots,t_v$. Hence if we violate an outside implication $r_1r_2\cdots r_{p-1}\to_0 t_i$ for some $i\in\{1,2,\dots,v\}$, we will violate $\max\{1,v-p+1\}$ remaining outside implications and hence at least $\frac{1}{p}$ of the outside implications from $r_1r_2\cdots r_{p-1}$. Hence there will be at most $(t-1)p\binom{m}{p-1}$ columns violating outside implications of $\mathrm{Imp}''_0(A)$. Thus the total number of columns in A is $O(m^{p-1})$. ∎

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Richard Anstee

Mathematics Department
The University of British Columbia
Vancouver, B.C.
V6T 1Z2
Canada
anstee@math.ubc.ca

Attila Sali

Alfréd Rényi Institute of Mathematics
The Hungarian Academy of Sciences
Budapest, P.O.B. 127
H-1364
Hungary
sali@renyi.hu